1. Given the differential equation: \( \frac{dy}{dx} = \frac{x}{y^2} \)

   a. Sketch the slope field for the points: (1,±1), (2,±1), (−1, ±1), and (0,±1).

   b. Find the general solution for the given differential equation.

   c. Find the solution of this differential equation that satisfies the initial condition \( y(0) = 2 \).
1. Given the differential equation: \( \frac{dy}{dx} = \frac{x}{y^2} \)

   a. Sketch the slope field for the points: \((1, \pm 1), (2, \pm 1), (-1, \pm 1), \) and \((0, \pm 1)\).

   ![Slope field diagram]

   +1 slopes of 0 along \( y \)-axis
   +1 relative steepness and correct slant of others

   b. Find the general solution for the given differential equation.

   \[
   \frac{dy}{dx} = \frac{x}{y^2} \\
   y^2 \, dy = x \, dx \\
   \int y^2 \, dy = \int x \, dx \\
   \frac{1}{3} y^3 = \frac{1}{2} x^2 + C \\
   y^3 = \frac{3}{2} x^2 + k \\
   y = \sqrt[3]{\frac{3}{2} x^2 + k}
   \]

   +2 correct antiderivatives
   +1 correct use of constant of integration
   +1 correct solution for \( y \)

   c. Find the solution of this differential equation that satisfies the initial condition \( y(0) = 2 \)

   \[
   2 = \sqrt[3]{\frac{3(0)^2}{2} + k} \\
   8 = k \\
   y = \sqrt[3]{\frac{3x^2}{2} + 8}
   \]

   +1 correct substitution of point \((0, 2)\)
   +1 correct solution
2. Given the differential equation: \( \frac{dy}{dx} = \frac{1}{xy} \).

a. Sketch the slope field for the points (1,±1), (2,±1), and (-1,±1).

b. Find the general solution for the given differential equation.

c. Find the solution of this differential equation that satisfies the initial condition \( y(1) = 2 \).
2. Given the differential equation: \( \frac{dy}{dx} = \frac{1}{xy} \).

a. Sketch the slope field for the points (1,±1), (2,±1), and (-1,±1).

b. Find the general solution for the given differential equation.

\[
\frac{dy}{dx} = \frac{1}{xy} \\
y \, dy = \frac{1}{x} \, dx \\
\int y \, dy = \int \frac{1}{x} \, dx \\
\frac{1}{2} \, y^2 = \ln|x| + C \\
y^2 = 2 \ln|x| + k \\
y = \pm \sqrt{2 \ln|x| + k}
\]

c. Find the solution of this differential equation that satisfies the initial condition \( y(1) = 2 \).

\[
2 = \sqrt{2 \ln|1| + k} \\
4 = k \\
y = \sqrt{2 \ln|x| + 4}
\]
3. **Given the differential equation** \( y' = \frac{2x}{y} \).

   a. **Use Euler’s Method** to find the first three approximations of the particular solution passing through \((1,2)\). Use a step of \( h = 0.2 \).

   b. **Find the general solution** of the differential equation.

   c. **Find the particular solution** of the differential equation that passes through \((1,2)\).
3. Given the differential equation \( y' = \frac{2x}{y} \)

<p>| | | |</p>
<table>
<thead>
<tr>
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</table>
| a. | Use Euler’s Method to find the first three approximations of the particular solution passing through \((1, 2)\). Use a step of \( h = 0.2 \) | +1 for 2.2  
+1 for 2.418  
+1 for 2.650 |
|   | \( y_1 = 2 + 0.2 \left( \frac{2(1)}{2} \right) = 2.2 \) |   |
|   | \( y_2 = 2.2 + 0.2 \left( \frac{2(1.2)}{2.2} \right) = 2.418 \) |   |
|   | \( y_3 = 2.418 + 0.2 \left( \frac{2(1.4)}{2.418} \right) = 2.650 \) |   |
| b. | Find the general solution of the differential equation. | +1: separates variables  
+2: correct antiderivatives  
+1: constant of integration |
|   | \( \frac{dy}{dx} = \frac{2x}{y} \) |   |
|   | \( y \, dy = 2x \, dx \) |   |
|   | \( \int y \, dy = \int 2x \, dx \) |   |
|   | \( \frac{1}{2} y^2 = x^2 + C \) |   |
|   | \( y^2 = 2x^2 + k \) |   |
|   | \( y = \pm \sqrt{2x^2 + k} \) |   |
| c. | Find the particular solution of the differential equation that passes through \((1, 2)\). | +1: uses initial condition  
+1: solves for \( y \) |
|   | \( 2 = \sqrt{2(1)^2 + k} \) |   |
|   | \( 4 = 2 + k \) |   |
|   | \( k = 2 \) |   |
|   | \( y = \sqrt{2x^2 + 2} \) |   |
4. Given the differential equation: \( y' = xy \)

   a. Use Euler’s Method to find the first three approximations of the particular solution passing through (1, 1). Use a step of \( h = 0.1 \).

   b. Find the general solution of the differential equation.

   c. Find the particular solution of the differential equation that passes through (1, 1).
4. Given the differential equation: \( y' = xy \)

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<table>
<thead>
<tr>
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</thead>
<tbody>
<tr>
<td><strong>a.</strong> Use Euler's Method to find the first three approximations of the particular solution passing through ((1, 1)). Use a step of ( h = 0.1 ).</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( y_1 = 1 + 0.1[1(1)] = 1.1 )</td>
<td>+1 for 1.1</td>
<td></td>
</tr>
<tr>
<td>( y_2 = 1.1 + 0.1[(1.1)(1.1)] = 1.221 )</td>
<td>+1 for 1.221</td>
<td></td>
</tr>
<tr>
<td>( y_3 = 1.221 + 0.1[(1.2)(1.221)] = 1.36752 )</td>
<td>+1 for 1.368</td>
<td></td>
</tr>
<tr>
<td><strong>b.</strong> Find the general solution of the differential equation.</td>
<td>+1: separates variables</td>
<td>+1: constant of integration</td>
</tr>
<tr>
<td>( \frac{dy}{dx} = xy )</td>
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<tr>
<td>( \frac{dy}{y} = x , dx )</td>
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<tr>
<td>( \ln</td>
<td>y</td>
<td>= \frac{1}{2} x^2 + C )</td>
</tr>
<tr>
<td>( y = e^{\frac{x^2}{2}} \cdot e^C )</td>
<td></td>
<td></td>
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<tr>
<td>( y = Ae^{\frac{x^2}{2}} )</td>
<td></td>
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<tr>
<td><strong>c.</strong> Find the particular solution of the differential equation that passes through ((1, 1)).</td>
<td>+1: uses initial condition</td>
<td>+1: solves for ( y )</td>
</tr>
<tr>
<td>( 1 = Ae^{\frac{1}{2}} )</td>
<td></td>
<td></td>
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<tr>
<td>( A = e^{-\frac{1}{2}} )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( y = e^{\frac{x^2}{2}-\frac{1}{2}} )</td>
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</tbody>
</table>
5. The number of bacteria in a culture is increasing according to the law of exponential growth. There are 150 bacteria in the culture after 3 hours and 400 bacteria after 5 hours.

a. Write an exponential growth model for the bacteria population.

b. Identify the initial population and the continuous growth rate.

c. Use logarithms to determine after how many hours the bacteria count will be approximately 15,000
5. The number of bacteria in a culture is increasing according to the law of exponential growth. There are 150 bacteria in the culture after 3 hours and 400 bacteria after 5 hours.

<table>
<thead>
<tr>
<th>a. Write an exponential growth model for the bacteria population.</th>
</tr>
</thead>
<tbody>
<tr>
<td>(150 = Ce^{3k}; \quad 400 = Ce^{5k})</td>
</tr>
<tr>
<td>Therefore, (400 = 150e^{-3k} \cdot e^{5k})</td>
</tr>
<tr>
<td>(\frac{8}{3} = e^{2k})</td>
</tr>
<tr>
<td>(\ln\frac{8}{3} = 2k)</td>
</tr>
<tr>
<td>(k = \frac{\ln(\frac{8}{3})}{2})</td>
</tr>
<tr>
<td>(150 = Ce^{-\frac{\ln(\frac{8}{3})}{2}})</td>
</tr>
<tr>
<td>So, (C = 34.447)</td>
</tr>
<tr>
<td>(y = 34.447e^{0.4904t})</td>
</tr>
<tr>
<td>+1 correct value for (k)</td>
</tr>
<tr>
<td>+1 correct value for (C)</td>
</tr>
<tr>
<td>+1 exponential equation for (y)</td>
</tr>
</tbody>
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<tr>
<th>b. Identify the initial population and the continuous growth rate.</th>
</tr>
</thead>
<tbody>
<tr>
<td>The initial population is approximately 34.447 and the continuous growth rate is 49.04%.</td>
</tr>
<tr>
<td>+1 initial population 34.447</td>
</tr>
<tr>
<td>+1 growth rate 49.04%</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>c. Use logarithms to determine after how many hours the bacteria count will be approximately 15,000?</th>
</tr>
</thead>
<tbody>
<tr>
<td>(15000 = 34.447e^{0.4904t})</td>
</tr>
<tr>
<td>(435.452 = e^{0.4904t})</td>
</tr>
<tr>
<td>(\ln 435.452 = 0.4904t)</td>
</tr>
<tr>
<td>(t = 12.391) hours</td>
</tr>
<tr>
<td>+1 set (y = 15,000)</td>
</tr>
<tr>
<td>+2 correct use of logs</td>
</tr>
<tr>
<td>+1 correct answer</td>
</tr>
</tbody>
</table>
6. The number of bacteria in a culture is increasing according to the law of exponential growth. There are 7,768 bacteria in the culture after 5 hours and 26,576 bacteria after 8 hours.

a. Write an exponential growth model for the bacteria population.

b. Identify the initial population and continuous growth rate.

c. After how many hours will the bacteria count be approximately 25,000? Use logarithms to find your answer.
6. The number of bacteria in a culture is increasing according to the law of exponential growth. There are 7,768 bacteria in the culture after 5 hours and 26,576 bacteria after 8 hours.

a. Write an exponential growth model for the bacteria population.

\[ 7768 = Ce^{5k}; \quad 26576 = Ce^{8k} \]

Therefore, \( 26576 = 7768e^{-5k} \cdot e^{8k} \)

\[ 3.4212 = e^{3k} \]

\[ \ln 3.4212 = 3k; \quad so, k = .4099986 \]

\[ 7768 = Ce^{.4099986(5)} \]

\[ C = 1000 \]

\[ y = 1000e^{0.41t} \]

b. Identify the initial population and continuous growth rate.

The initial population is approximately 1000 bacteria and the continuous growth rate is 41%.

+1: initial population is 1000

+1: growth rate is 41%

+1: correct value for \( k \)

+1: correct value for \( C \)

+1 exponential equation for \( y \)

+1: correct use of logs

+1 correct answer

c. After how many hours will the bacteria count be approximately 25,000? Use logarithms to find your answer.

\[ 25000 = 1000e^{41t} \]

\[ 25 = e^{41t} \]

\[ \ln 25 = .41t \]

\[ t = 7.851 \text{ hours} \]
7. The rate of the spread of an epidemic is jointly proportional to the number of infected people and the number of uninfected people. In an isolated town of 4000 inhabitants, 160 people have a disease at the beginning of the week and 1200 have it a week later.

a. Write a model for this population.

b. How many will be infected after 10 days?

c. How long does it take for 80% of the population to become infected?
### 7.
The rate of the spread of an epidemic is jointly proportional to the number of infected people and the number of uninfected people. In an isolated town of 4000 inhabitants, 160 people have a disease at the beginning of the week and 1200 have it a week later.

#### a. Write a model for this population.

\[
\frac{dy}{dx} = k(y)(4000 - y) = K\left(y - \frac{y}{4000}\right)
\]

\[
\frac{dy}{y(1 - \frac{y}{4000})} = K\, dt
\]

\[
\int \left(\frac{1}{y} + \frac{1}{4000-y}\right)\,dy = \int K\, dt
\]

\[
\ln\left|\frac{4000-y}{y}\right| = -Kt - C \Rightarrow \frac{4000-y}{y} = Ae^{-Kt}
\]

\[
y = \frac{4000}{1+ Ae^{-Kt}}\quad y(160) = \frac{4000}{1+ Ae^{-K(10)}} \Rightarrow A = 24
\]

\[
y = \frac{4000}{1+ 24e^{-0.332965t}}
\]

\[
y(1200) = \frac{4000}{1+ 24e^{-0.332965(7)}} \Rightarrow K = 0.332965.
\]

#### b. How many will be infected after 10 days?

\[
y = \frac{4000}{1+ 24e^{-0.332965(10)}}
\]

Approximately 2151 people will be infected after 10 days.

#### c. How long does it take for 80% of the population to become infected?

\[
y = \frac{4000}{1+ 24e^{-0.332965t}} = 3200
\]

\[
1 + 24e^{-0.332965t} = 1.25\quad e^{-0.332965t} = \frac{1}{24}\quad t \approx 13.7
\]

It will take approximately 14 days for 80% of the population to become infected.
8. Given the logistic equation: \( \frac{dy}{dt} = 2y(1 - \frac{y}{5}) \)

a. Determine the general solution of this equation.

b. Find the particular solution satisfying \( y(0) = 3 \).

c. Evaluate \( \lim_{t \to \infty} y \) and explain its meaning.
8. Given the logistic equation: \( \frac{dy}{dt} = 2y(1 - \frac{y}{5}) \)

a. Determine the general solution of this equation.

\[
\frac{dy}{y\left(1 - \frac{y}{5}\right)} = 2dt
\]

\[
\int \frac{dy}{y\left(1 - \frac{y}{5}\right)} = \int 2dt
\]

\[
\int \left(\frac{1}{y} + \frac{1}{5-y}\right) dy = \int 2dt
\]

\[
\ln\left|\frac{5-y}{y}\right| = -2t - C
\]

\[
\frac{5-y}{y} = e^{-2t-C} \Rightarrow \frac{5-y}{y} = Be^{-2t}
\]

\[
y = \frac{5}{1 + Be^{-2t}}
\]

b. Find the particular solution satisfying \( y(0) = 3 \).\n
\[
3 = \frac{5}{1 + Be^{-2(0)}}
\]

\[
3(1 + B) = 5
\]

\[
B = \frac{2}{3}
\]

\[
y = \frac{5}{1 + \frac{2}{3}e^{-2t}}
\]

\[\text{c. Evaluate } \lim_{t \to \infty} y \text{ and explain its meaning.}\]

Since \( e^{-2t} \to 0 \text{ as } t \to \infty \), \( \lim_{t \to \infty} \frac{5}{1 + \frac{2}{3}e^{-2t}} = 5 \).

This represents the carrying capacity of the population modeled by the given logistic differential equation.
9. A 200-gallon tank contains 100 gallons of water with a sugar concentration of 0.1 pounds per gallon. Water, with a sugar concentration of 0.4 pounds per gallon, flows into the tank at a rate of 20 gallons per minute. Assume the mixture is mixed instantly and water is pumped out at a rate of 10 gallons per minute. Let $y(t)$ be the amount of sugar in the tank at time $t$.

a. Set up the differential equation which models the given information.

b. Solve the differential equation for $y(t)$.

c. What is the amount of sugar in the tank when it overflows?
9. A 200-gallon tank contains 100 gallons of water with a sugar concentration of 0.1 pounds per gallon. Water, with a sugar concentration of 0.4 pounds per gallon, flows into the tank at a rate of 20 gallons per minute. Assume the mixture is mixed instantly and water is pumped out at a rate of 10 gallons per minute. Let \( y(t) \) be the amount of sugar in the tank at time \( t \).

a. Set up the differential equation which models the given information.

Rate of sugar coming in =
\[
0 \cdot \frac{4}{\text{gal}} \left( \frac{20}{\text{gal/min}} \right) = \frac{8}{\text{lb/min}}
\]

Rate of sugar going out =
\[
\left( \frac{y}{100+10t} \right) \left( \frac{10}{\text{gal/min}} \right) = \frac{y}{10+t} \text{ lb/min}
\]

Therefore,
\[
\frac{dy}{dt} = 8 - \frac{y}{10+t}
\]

b. Solve the differential equation for \( y(t) \).

\[
\alpha(t) = e^{\frac{1}{t+10}} = e^{\ln(t+10)} = t + 10
\]

\[
\Rightarrow y(t) = \frac{1}{t+10} \int (t+10)(8)dt
\]

\[
y(t) = 4t + 40 - \frac{300}{t+10}
\]

+2: rate of sugar coming in of 8 lb/min
+2: rate of sugar going out of \( \frac{y}{10+t} \) lb/min

+2: integrating factor of \( t+10 \)
+1: solution for \( y(t) \)

+1: setting 100 + 10t = 200
+1: answer

c. What is the amount of sugar in the tank when it overflows?

Since the tank contains 100 + 10t gallons at time \( t \), when the tank overflows 100 + 10t = 200. Therefore, the tank overflows in 10 minutes.

\[
y(10) = 4(10) + 40 - \frac{300}{10+10}
\]

\[
y(10) = 65
\]

65 pounds of sugar are in the tank when it overflows.
10. A 300-gallon tank contains 200 gallons of water with a salt concentration of 0.2 pounds per gallon. Water, with a salt concentration of 0.3 pounds per gallon, flows into the tank at a rate of 20 gallons per minute. Assume that the mixture is mixed instantly and water is pumped out at a rate of 10 gallons per minute.

a. Set up the differential equation which models this information.

b. Solve the differential equation for \( y(t) \).

c. What is the amount of sugar in the tank when it overflows?
10. A 300-gallon tank contains 200 gallons of water with a salt concentration of 0.2 pounds per gallon. Water, with a salt concentration of 0.3 pounds per gallon, flows into the tank at a rate of 20 gallons per minute. Assume that the mixture is mixed instantly and water is pumped out at a rate of 10 gallons per minute.

a. Set up the differential equation which models this information.

\[
\text{Rate of salt coming in} = \left(0.3 \frac{\text{lb}}{\text{gal}}\right)\left(20 \frac{\text{gal}}{\text{min}}\right) = 6 \frac{\text{lb}}{\text{min}}
\]

\[
\text{Rate of salt going out} = \left(\frac{y}{200+10t} \frac{\text{lb}}{\text{gal}}\right)\left(10 \frac{\text{gal}}{\text{min}}\right) = \frac{y}{20+t} \frac{\text{lb}}{\text{min}}
\]

Therefore, \( \frac{dy}{dt} = 6 - \frac{y}{t+20} \)

b. Solve the differential equation for \( y(t) \).

\[
\frac{dy}{dt} + \left(\frac{1}{t+20}\right)y = 6
\]

\[
\alpha(t) = e^{\int \frac{1}{t+20} dt} = e^{\ln(t+20)} = t+20
\]

\[
\Rightarrow y(t) = \frac{1}{t+20} \int (t+20)(6)dt
\]

Therefore, \( y(t) = 3t + 60 - \frac{400}{t+20} \)

\[+2: \text{integrating factor of } t+20\]

\[+1: \text{solution for } y(t)\]

c. What is the amount of salt in the tank when it overflows?

Since the tank contains 200 + 10t gallons at time t, when it overflows 200 + 10t = 300. Therefore, the tank overflows in 10 minutes.

\[
y(10) = 3(10) + 60 - \frac{400}{10+20}
\]

\[
y(10) = \frac{230}{3}
\]

\[
\frac{230}{3} \text{ pounds of salt are in the tank when it overflows}\]

\[+1: \text{setting } 200+10t=300\]

\[+1: \text{answer}\]